Proof Of The Infinite Sum Of A Geometric Series

1 Introduction

Before the proof begins, first we will define what sort of series is being considered. The sums in question are of the form

$$\sum_{i=1}^{\infty} ar^{i-1}$$

As is clear from the series, a denotes the initial value of the sequence (a_n) which is being summed ($a_n = ar^{n+1}$) since $a_1 = ar^0 = a$, and r is the ratio between successive terms in the sequence ($\frac{a_{n+1}}{a_n} = r$).

As an example, consider the sequence $4, 12, 36, \ldots$. a would be 4, since the first value in the sequence is 4, and the ratio between any two successive terms is $r = 3$.

2 Finite (Partial) Sum

The nth partial sum of a series is the sum of the first n terms in the sequence. Therefore, we can express the nth partial sum of a sequence, (a_n), as

$$\sum_{i=1}^{n} a_n$$

Our goal is to express the nth partial sum of a geometric series. To do so, for ease of notation, let $S_n := \sum_{i=1}^{n} ar^{i-1}$. By definition, $S_n = a + ar + ar^2 + ... + ar^{n-1}$. If we subtract the first term and divide by r, we get the $n-1$th partial sum ($\frac{S_n - a}{r} = a + ar + ... + ar^{n-2} = S_{n-1}$). We also get the $n-1$th partial sum if we instead simply subtract the final term of the series ($S_n - ar^{n-1} = a + ar + ... + ar^{n-2} = S_{n-1}$). Because these are equivalent, we can say that

$$\frac{S_n - a}{r} = S_n - ar^{n-1}$$

$$\implies S_n - a = rS_n - ar^n$$
$$\implies S_n - rS_n = a - ar^n$$
$$\implies S_n(1-r) = a(1-r^n)$$
$$\implies S_n = \frac{a(1-r^n)}{1-r}$$

This gives us our desired result of a formula for the nth partial sum, where

$$\sum_{i=1}^{n} ar^{i-1} = \frac{a(1-r^n)}{1-r}$$

This case only works if $r \neq 1$; however, it is trivial to find the nth partial sum in this case since all entries of the sequence are a, meaning that the nth partial sum is equal to an.

For one example, consider the example sequence from the introduction. The third partial sum of the series is equal to $S_3 = 4 + 12 + 36 = 52$. Substituting the values of a, r, n into the formula we derived gives $S_3 = \frac{4(1-3^3)}{1-3} = \frac{4(-26)}{-2} = 2(26) = 52$.

3 Infinite Sum

Before we derive a formula for the infinite series, it is important to first consider when the series converges. The method that will be used here is the Ratio Test. The test states that if $\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = L$ then the following is true:

- $L < 1$, the series converges absolutely
- $L > 1$, the series diverges
- $L = 1$, the test is inconclusive

Applying the Ratio Test to our geometric series gives

$$\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \frac{|ar^n|}{|ar^{n-1}|}$$

$$\implies \lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = |r| = L$$

Since $L = |r|$, we know that geometric series converges absolutely if $|r| < 1$, diverges if $|r| > 1$. The case of $L = 1$ occurs only when $r = \pm 1$ and if $r = 1$ then, unless $a = 0$, the series diverges. Because of this, we will only consider the case where $|r| < 1$.

The infinite series can be defined as the limit of the partial sums. That is to say that

$$\sum_{i=1}^{\infty} ar^{i-1} := \lim_{n\to\infty} \sum_{i=1}^{n} ar^{i-1}$$

Since we already know an expression for the partial sum, we can alternatively write this as

$$\sum_{i=1}^{\infty} ar^{i-1} = \lim_{n\to\infty} \frac{a(1-r^n)}{1-r}$$

Since we are only considering the case of $|r| < 1$, we know that the $\lim_{n\to\infty} r^n = 0$.

$$\implies \sum_{i=1}^{\infty} ar^{i-1} = \frac{a(1-0)}{1-r} = \frac{a}{1-r}$$

Which is the final answer.