Integral Test For Series

1 Introduction

The Integral Test for series is a test for checking if a series converges by comparing it to a continuous function. The test states that:

Suppose $f(x)$ is a continuous, positive, decreasing function on an interval $[m, \infty)$, and that $f(n) = a_n$, then:

- If $\int_m^\infty f(x)dx$ is convergent then $\sum_{n=m}^\infty a_n$ is also convergent
- If $\int_m^\infty f(x)dx$ is divergent then $\sum_{n=m}^\infty a_n$ is also divergent

Further, it is true that, for any $N \geq m$, the series evaluated from N to ∞ is bounded by:

$$\int_{N+1}^\infty f(x)dx \leq \sum_{n=N}^\infty a_n \leq \int_N^\infty f(x)dx$$

More generally (here m is an integer greater than or equal to the m from which f is defined),

$$\int_{m+1}^{n+1} f(x)dx \leq \sum_{k=m+1}^{n} a_k \leq \int_m^n f(x)dx$$

2 Proof

Note that since f is decreasing, $f(n+1) < \int_n^{n+1} f < f(n)$, and further $\int_m^\infty f = \lim_{t\to\infty} \int_m^t f = \sum_{n=m}^\infty \int_n^{n+1} f$. The convergence of the improper integral is dependent on the convergent of the series.

First, we will consider the left side of the inequality, namely that $f(n+1) < \int_n^{n+1} f$. This means that:

$$a_{n+1} < \int_n^{n+1} f$$

$$\implies \sum_{n=m}^{\infty} a_{n+1} \leq \sum_{n=m}^{\infty} \int_n^{n+1} f = \int_m^{\infty} f$$

By the Comparison Test, $\sum a_{n+1}$ converges if $\sum_{n=m}^{\infty} \int_n^{n+1} f$ converges. Since the sum is equal to $\int_m^{\infty} f$, $\sum a_{n+1}$ converges if $\int_m^{\infty} f$ converges. By the Shift Property of series, this also means that $\sum a_n$ converges if $\int_m^{\infty} f$ converges.

Now, for the right side ($\int_n^{n+1} f < f(n)$), since $f(n) = a_n$,

$$\int_n^{n+1} f < a_n$$

$$\implies \int_m^{\infty} f = \sum_{n=m}^{\infty} \int_n^{n+1} f \leq \sum_{n=m}^{\infty} a_n$$

By the Comparison Test, $\sum_{n=m}^{\infty} a_n$ diverges is $\int_m^{\infty} f$ diverges.

For the bounds of the series, the lower bound is simple to prove. Since $\int_n^{n+1} f < a_n$, clearly

$$\sum_{k=m+1}^{n} \int_k^{k+1} = \int_{m+1}^{n+1} f \leq \sum_{k=m+1}^{n} a_n$$

The upper bound is also simple to prove. As stated before, $a_{n+1} < \int_n^{n+1} f$, and so $a_n < \int_{n-1}^{n} f$, therefore

$$\sum_{k=m+1}^{n} a_n \leq \int_m^n f = \sum_{k=m+1}^{n} \int_{k-1}^{k} f$$

$$\implies \int_{m+1}^{n+1} f \leq \sum_{k=m+1}^{n} a_n \leq \int_m^n f$$

The infinite bounds follow simply by taking the proper integrals to infinity.